SENSITIVITY OF EQUILIBRIUM IN ATOMIC CONGESTION GAMES WITH PLAYER-SPECIFIC COST FUNCTIONS

Frédéric Meunier, Thomas Pradeau

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FRÉDÉRIC MEUNIER AND THOMAS PRADEAU

Abstract. Consider an atomic splittable congestion game played on a parallel-link graph with player-specific cost functions. Richman and Shimkin proved in 2007 that the equilibrium is unique when the cost functions are continuous, increasing, and strictly convex. It allows to define a function $\vec{\boldsymbol{e}}(\cdot)$ mapping any demand vector to the unique corresponding equilibrium. The general question we address in this paper is about the behavior of $\vec{\boldsymbol{e}}(\cdot)$: how does the equilibrium change when the demand vector changes?

By standard arguments regarding the solutions of variational inequalities, we prove that $\vec{\boldsymbol{e}}(\cdot)$ is continuous. Our main results concern the case when there are only two players or only two arcs. We show that if the cost functions are twice continuously differentiable, $\vec{\boldsymbol{e}}(\cdot)$ is differentiable at any point on a neighborhood of which the supports of the player strategies remain constant. We are able to describe precisely what happens to the support of the strategies at equilibrium when a player transfers a part of his demand to another player with more demand. It allows to recover previous results for this kind of game regarding the impact of coalitions on the equilibrium and to discuss the impact of such transfers on the social cost.

We show moreover that most of these results do not hold when there are at least three players and three arcs.

1. Introduction

In many contexts, users share common resources to realize some tasks while being not coordinated. Examples of such resources are means of transportation, machines in a flexible manufacturing environment, or arcs in a telecommunication network. Congestion may appear on resources, leading to strategic behaviors. Game theory is a useful approach to understand and predict the behaviors of the users and the resulting congestion on the resources. The games arising in such a context, called *congestion games*, have been studied since the 50's, addressing many questions, such as the existence and the uniqueness of the Nash equilibrium, the way to compute it and related complexity questions, or its efficiency with respect to the social optimum (via the so-called "Price of Anarchy").

One stream of questions is about the *sensitivity analysis*, defined as the evaluation of the impact of the input (graph, cost functions and demands) on the equilibrium. In practice, these analyses are used for designing networks, estimating origin-destination matrices, or fixing pricing rules. Such analyses have been mainly applied to nonatomic congestion games and formulas have been designed in order to perform the sensitivity analysis, see Tobin and Friesz [1988], Qiu and Magnanti [1989], Bell and Iida [1997]. *Nonatomic* means that there is a continuous set of users, each with a negligible impact on the congestion. Taking a more theoretical point of view, Hall [1978] proved that the equilibrium flows of nonatomic games played on a network is continuous with respect to the demand when all users have the same cost functions. This result have been extended for cost functions depending on the flows on all arcs of the network by Dafermos and Nagurney [1984]. A more general study, concerning in particular the differentiability, has been made by Patriksson and Rockafellar [2003] and Patriksson [2004]. The latter gave a characterization for the existence of a directional derivative of the equilibrium flow with respect to the demand. Josefsson and Patriksson

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[2007] showed that while equilibrium costs are directionally differentiable, this does not hold for the flows.

A natural intuition would be that an increase of the demand gives an increase of the equilibrium cost. Hall [1978] proved that this result is true for two-terminal graphs when players have the same cost functions, and Lin et al. [2011] gave an alternate combinatorial proof of this result. However this intuition is false in general, as noted for example by Fisk [1979]. Dafermos and Nagurney [1984] proved that an "average" total cost will necessarily increase. More recently, Englert et al. [2008] proved that there are networks for which a slight increase of the total demand changes the strategies of all users. This "microscopic" instability, in the sense that the total flow on each arc does not change too much, happens in particular for the class of *generalized Braess graphs* introduced by Roughgarden [2006].

Our purpose is to make a sensitivity analysis for *atomic splittable* games. We consider a finite set of non-negligible players having a stock, the *demand*, to divide among different resources. For example, a freight company may have to choose between several means of transportation. This situation can be modeled by an atomic game on a network with parallel arcs, each arc representing a resource. This kind of games have been extensively studied, see for example Orda et al. [1993], Altman et al. [2002], Richman and Shimkin [2007], Bhaskar et al. [2009, 2010]. Further results and extensions on Nash equilibria in this context can be found in Gairing et al. [2006], Harks [2008], Cominetti et al. [2009]. These works consider mostly the case when every player is impacted in the same way by the congestion.

In this paper, we study the sensitivity of the equilibrium in atomic splittable games on parallel-arcs graphs with player-specific cost functions. Questions like the continuity of the equilibrium and its differentiability are addressed, as well as questions about the consequence of a partial transfer of demand from a player to another one with a higher demand. The impact of making coalitions is a special case of this latter question.

2. Model and main results

2.1. **Model.** We are given a two-terminal graph with a set A of parallel arcs and $K \geq 2$ players identified with the integers $1, \ldots, K$. Throughout the paper, the set $\{1, \ldots, K\}$ is denoted $[K]$. We consider the model of atomic splittable games: each player k has a demand $d^k \in \mathbb{R}_+$ and cost functions $c_a^k : \mathbb{R}_+ \to \mathbb{R}_+$, assumed to be increasing, differentiable, and strictly convex. A *feasible strategy* for the player k is an element $\boldsymbol{x}^k = (x_a^k)_{a \in A}$ of $\mathbb{R}_+^A$ such that $\sum_{a \in A} x_a^k = d^k$. The quantity x_a^k is then the *flow of player* k on the arc a. When each player has chosen a strategy, the *total flow* on arc a is the quantity $x_a = \sum_{k=1}^K x_a^k$ and the *support* of the strategy is the set of arcs "used" by this player

$$\operatorname{supp}(\boldsymbol{x}^k) = \{a \in A : \; x_a^k > 0\}.$$

The cost supported by player k is $\sum_{a \in A} c_a^k(x_a)$. A vector $\vec{\boldsymbol{x}} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)_{a \in A}$ of feasible strategies is an *equilibrium* if for every player k

$$\sum_{a \in A} x_a^k c_a^k(x_a) = \min_{(y_a^k) \in \mathbb{R}_+^A : \; \sum_a y_a^k = d^k} \sum_{a \in A} y_a^k c_a^k(y_a).$$

2.2. **Properties.** The following characterization of the equilibrium is standard and has been used for instance in Haurie and Marcotte [1985].

Proposition 1. *The vector $\vec{\boldsymbol{x}}$ is an equilibrium if and only if, for all k, the vector $\boldsymbol{x}^k$ is a feasible strategy for player k and*

$$\sum_{a \in A} (c_a^k(x_a) + x_a^k {c_a^k}'(x_a))(y_a^k - x_a^k) \geq 0, \quad \textit{for any other feasible strategy } \boldsymbol{y}^k.$$

The following proposition is an alternate and again standard characterization of the equilibrium.

Proposition 2. *The vector $\vec{\boldsymbol{x}}$ is an equilibrium if and only if, for all k, the vector $\boldsymbol{x}^k$ is a feasible strategy for player k and $c_a^k(x_a) + x_a^k c_a^{k\prime}(x_a) = \pi^k$ for all $a \in \mathrm{supp}(\boldsymbol{x}^k)$, where*

$$\pi^k = \min_{a \in A} \left(c_a^k(x_a) + x_a^k c_a^{k\prime}(x_a) \right).$$

$c_a^k(x_a) + x_a^k c_a^{k\prime}(x_a)$ is the marginal cost of arc a for player k. Proposition 2 states thus that at equilibrium, the marginal costs of the arcs in the support of every player are all equal.

For the games considered in this paper, an equilibrium always exists, see Rosen [1965], Orda et al. [1993]. Furthermore, the graph belongs to the class of nearly-parallel graphs and thus the equilibrium is unique [Richman and Shimkin, 2007].

2.3. **Results.** The uniqueness of equilibrium allows to define for each player k the map

$$\boldsymbol{e}^k : \mathbb{R}_+^K \longrightarrow \mathbb{R}_+^A$$

by defining $\boldsymbol{e}^k(\boldsymbol{d})$ to be the strategy of player k at equilibrium when the demands are given by the components of $\boldsymbol{d} = (d^1, \ldots, d^K)$. Define

$$\vec{\boldsymbol{e}} : \boldsymbol{d} \in \mathbb{R}_+^K \longmapsto (\boldsymbol{e}^1(\boldsymbol{d}), \ldots, \boldsymbol{e}^K(\boldsymbol{d})) \in (\mathbb{R}_+^A)^K.$$

It is a continuous map, see Proposition 3 of Section 3.

A *demand transfer* is a part $\delta \in [0, d^i]$ removed of the demand of a player i and added to the demand of a player j, the demand of the other players remaining the same. The new demand of player i is then $d^i - \delta$ and the new demand of player j is then $d^j + \delta$.

Theorem 1. *Suppose that $|A| = 2$ or $K = 2$, and that the cost functions are twice continuously differentiable. Let $\boldsymbol{d}_0 \in \mathbb{R}_+^K$ be a demand vector such that for any pair of players, a sufficiently small nonzero demand transfer does not modify the supports of the players. Then $\vec{\boldsymbol{e}}(\cdot)$ is differentiable at $\boldsymbol{d}_0$.*

Generically, all $\boldsymbol{d}_0 \in \mathbb{R}_+^K$ satisfy this condition (it is for instance a consequence of Corollary 1 of Section 4.1). We do not know whether this theorem still holds when $|A| \geq 3$ and $K \geq 3$. However, we cannot expect $\vec{\boldsymbol{e}}(\cdot)$ to be differentiable everywhere, even if there is only one player, see Section 6.

Our two other main results are about the case when the players i and j involved in the transfer described above are such that $d^i \leq d^j$.

Theorem 2. *Suppose that $|A| = 2$ or $K = 2$. If a player i transfers a part of his demand to a player j with $d^i \leq d^j$, then, on each arc, the flow of player i (resp. player j) decreases (resp. increases) or remains constant equal to zero.*

The *social cost* is defined by

$$Q(\vec{\boldsymbol{x}}) = \sum_{k=1}^{K} \sum_{a \in A} x_a^k c_a^k(x_a).$$

It is the cost experienced by all players together.

Theorem 3. *Suppose that $c_a^1 = \cdots = c_a^K$ for all $a \in A$ (we are no longer in the player-specific cost setting). Suppose moreover that at least one of the following conditions is satisfied.*

- $|A| = 2$
- $K = 2$ *and the cost functions are twice continuously differentiable .*

Then if a player i transfers a part of his demand to a player j with $d^i \leq d^j$, the social cost at equilibrium decreases or remains constant.

A consequence of this theorem is that if the total demand $\sum_k d^k = d$ is fixed, the equilibrium with highest social cost is obtained when all players have the same demand equal to d/k. In the contrary, social cost is reduced when a monopole takes in charge the whole demand. Nevertheless, this is true under the conditions of the theorem. If these conditions are relaxed, there are situations where the conclusion of Theorem 3 does not hold, see Section 6.

Theorem 3 is a generalization of Theorem 3.23 of Wan [2012] when there are no nonatomic players. Wan [2012] proved that when there are two arcs the social cost at equilibrium cannot increase when two players merge, i.e. in our context when player i transfers all his demand to player j. The question whether our result remains valid with nonatomic players deserves future work.

Theorems 1, 2, and 3 are respectively proved in Sections 3, 4, and 5. We emphasize that the proof of Theorem 3 uses Theorems 1 and 2.

3. Regularity of equilibrium

3.1. **Continuity.** The following proposition holds for the model described in Section 2.1 in its full generality. It contrasts with Theorem 1, which requires the cost function to be twice continuously diffrrentiable.

Proposition 3. *The map $\vec{\boldsymbol{e}}(\cdot)$ is continuous.*

Proof. Let $\triangle = \{\boldsymbol{p} \in \mathbb{R}_+^A : \sum_{a\in A} p_a = 1\}$ be the $(|A|-1)$-dimensional simplex and $F : \triangle^K \times \mathbb{R}_+^K \to (\mathbb{R}_+^A)^K$ be defined by

$$F_a^k(\vec{\boldsymbol{p}}, \boldsymbol{d}) = \left[c_a^k \left(\sum_{\ell=1}^{K} p_a^\ell d^\ell \right) + p_a^k d^k c_a^{k\prime} \left(\sum_{\ell=1}^{K} p_a^\ell d^\ell \right) \right] d^k,$$

for every a and k. The application F is continuous in both variables.

According to Proposition 1, $\vec{\boldsymbol{e}}(\boldsymbol{d})$ is the equilibrium if and only if $e_a^k(\boldsymbol{d}) = p_a^k d^k$ for all a and k, where $\vec{\boldsymbol{p}}$ satisfies the variational inequality

$$(VI(\boldsymbol{d})) \qquad F(\vec{\boldsymbol{p}}, \boldsymbol{d}) \cdot (\vec{\boldsymbol{q}} - \vec{\boldsymbol{p}}) \geq 0 \quad \text{for any } \vec{\boldsymbol{q}} \in \triangle^K.$$

By a sequential argument, using the continuity of F and the compactness of $\triangle^K$, elementary calculations give that the solution of $(VI(\boldsymbol{d}))$ is continuous with respect to $\boldsymbol{d}$. □

3.2. **Proof of Theorem 1.** The strategies at equilibrium are completely determined by the supports of the players. This is due to the increasing assumption of the costs. When the supports are given, the equilibrium is characterized by equalities and then the inverse function theorem makes the job. This is the core idea of the proof of Theorem 1.

Lemma 1. *Suppose that the cost functions are twice continuously differentiable. Choose an $S^k \subseteq A$ for each player k. Define the application $H : \prod_{k=1}^K \mathbb{R}_+^{S^k} \times \mathbb{R}_+^K \to \prod_{k=1}^K \mathbb{R}^{S^k} \times \mathbb{R}^K$ by $H(\vec{\boldsymbol{y}}, \boldsymbol{\pi}) = (G(\vec{\boldsymbol{y}}, \boldsymbol{\pi}), D(\vec{\boldsymbol{y}}))$, where for $a \in S^k$, $\boldsymbol{y}^k \in S^k$, and $\boldsymbol{\pi} \in \mathbb{R}^K$, we have*

$$G_a^k(\vec{\boldsymbol{y}}, \boldsymbol{\pi}) = c_a^k(y_a) + y_a^k c_a^{k\prime}(y_a) - \pi^k \quad \text{and} \quad D^k(\vec{\boldsymbol{y}}) = \sum_{a\in S^k} y_a^k.$$

Let $\vec{\boldsymbol{z}}$ be such that $z_a > 0$ for every $a \in S^k$ and every player k, and let $\boldsymbol{\pi} \in \mathbb{R}^K$. Then H is continuously differentiable and invertible in a neighborhood of $(\vec{\boldsymbol{z}}, \boldsymbol{\pi})$, with a continuously differentiable inverse.

Proof. Since c_a is twice continuously differentiable for every a, the map H is continuously differentiable. We will show that the Jacobian matrix J of H at $(\vec{\boldsymbol{z}}, \boldsymbol{\pi})$ is nonsingular. The conclusion will then follow from the inverse function theorem.

We define $g_a^k = c_a^{k\prime}(z_a) + z_a^k c_a^{k\prime\prime}(z_a)$ for every $a \in S^k$ and $h_a^k = c_a^{k\prime}(z_a)$ for every $a \in A$. Note that because of the assumptions, they are positive when $a \in S^k$. A direct calculation gives the entries of J. For $k \in [K]$ and $a \in S^k$, we have

$$\frac{\partial G_a^k}{\partial y_b^\ell}(\vec{\boldsymbol{z}}, \boldsymbol{\pi}) = \begin{cases} g_a^k & \text{for } a = b,\ k \neq \ell \\ g_a^k + h_a^k & \text{for } a = b,\ k = \ell \\ 0 & \text{for } a \neq b. \end{cases}$$

$$\frac{\partial G_a^k}{\partial \pi^\ell}(\vec{\boldsymbol{z}}, \boldsymbol{\pi}) = \begin{cases} 0 & \text{for } k \neq \ell \\ -1 & \text{for } k = \ell \end{cases}$$

$$\frac{\partial D^k}{\partial y_b^\ell}(\vec{\boldsymbol{z}}, \boldsymbol{\pi}) = \begin{cases} 0 & \text{for } k \neq \ell \\ 1 & \text{for } k = \ell. \end{cases}$$

$$\frac{\partial D^k}{\partial \pi^\ell}(\vec{\boldsymbol{z}}, \boldsymbol{\pi}) = 0.$$

We prove now that J is nonsingular by showing that its kernel contains only the zero vector. We denote by K_a the set of players effectively using arc a:

$$K_a = \{k \in [K] : a \in S^k\}.$$

Let $(\vec{\boldsymbol{\lambda}}, \boldsymbol{\mu}) \in \prod_{k=1}^K \mathbb{R}^{S^k} \times \mathbb{R}^K$ be in the kernel of J. The equality $J(\vec{\boldsymbol{\lambda}}, \boldsymbol{\mu}) = \mathbf{0}$ can be written under the form

$$\begin{cases} M_a \boldsymbol{\lambda}_a = \boldsymbol{\mu}_a & \text{for } a \in A, \\ \displaystyle\sum_{a \in S^k} \lambda_a^k = 0 & \text{for } k \in [K]. \end{cases} \tag{1}$$

where $\boldsymbol{\lambda}_a = (\lambda_a^k)_{k \in K_a}$ and $\boldsymbol{\mu}_a = (\mu^k)_{k \in K_a}$, with $M_a = (m_{k,\ell})$ being the $|K_a| \times |K_a|$ matrix such that

$$m_{k,\ell} = \begin{cases} g_a^k & \text{for } k \neq \ell \\ g_a^k + h_a^k & \text{for } k = \ell. \end{cases}$$

It can be readily checked that M_a is a nonsingular matrix because the g_a^k and the h_a^k are positive.

In order to consider matrices with same dimension, we define $\vec{\boldsymbol{\Lambda}} \in \prod_{k=1}^K \mathbb{R}^A$ by $\Lambda_a^k = \lambda_a^k$ for $a \in K_a$ and 0 elsewhere. The system (1) is then equivalent to the following system:

$$\begin{cases} \boldsymbol{\Lambda}_a = M_a' \boldsymbol{\mu} & \text{for } a \in A, \\ \displaystyle\sum_{a \in A} \boldsymbol{\Lambda}_a = \mathbf{0}, \end{cases}$$

where M_a' is a $K \times K$ matrix obtained from M_a^{-1} by setting the missing coefficients to zero. These conditions imply that

$$\sum_{a \in A} \boldsymbol{\Lambda}_a = \left(\sum_{a \in A} M_a' \right) \boldsymbol{\mu} = \mathbf{0}.$$

A *Z-matrix* is a square matrix whose nondiagonal coefficients are nonpositive. According to a theorem by Minkowski [1900], if each column of a Z-matrix has a positive sum, then it is a nonsingular matrix. For more details, see [Berman and Plemmons, 1979, Chapter 6]. It can be checked that each M_a' is such a Z-matrix. Their sum is thus also such a Z-matrix, and is thus nonsingular. It gives $\boldsymbol{\mu} = \mathbf{0}$ and then $\boldsymbol{\Lambda}_a = \mathbf{0}$ for all $a \in A$. In particular, $\vec{\boldsymbol{\lambda}} = 0$. Therefore, J is nonsingular. □

Proof of Theorem 1. Let $\boldsymbol{d}_0$ be as in the statement. Define S^k to be $\text{supp}(\boldsymbol{e}^k(\boldsymbol{d}_0))$. Let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d}_0)$. There exists $\boldsymbol{\pi} \in \mathbb{R}_+^K$ such that

$$\left\{\begin{array}{rl} c_a^k(x_a) + x_a^k c_a^{k\prime}(x_a) - \pi^k = 0 & \text{for } k \in [K], a \in S^k \\ \sum_{a\in A} x_a^k = d_0^k & \text{for } k \in [K] \\ x_a^k = 0 & \text{for } k \in [K], a \notin S^k \\ c_a^k(x_a) + x_a^k c_a^{k\prime}(x_a) - \pi^k > 0 & \text{for } k \in [K], a \notin S^k. \end{array}\right.$$

Indeed, it is a consequence of Proposition 2, except the fact that the last inequality is strict. We prove now this latter fact.

Assume for a contradiction that there is a player k such that $c_a^k(x_a) + x_a^k c_a^{k\prime}(x_a) - \pi^k = 0$, while $a \notin S^k$. The continuity of $\vec{\boldsymbol{e}}$ and Corollary 2 (see Section 4) would then imply that a slight transfer of demand to player k would change the support of his strategy, which is in contradiction with the assumption regarding the non-modification of the support. Corollary 2 can be used since we have assumed that $|A| = 2$ or $K = 2$.

We can apply Lemma 1: there is a neighborhood V of $(\boldsymbol{0}, \boldsymbol{d}_0)$ in $\prod_{k\in[K]} \mathbb{R}^{S^k} \times \mathbb{R}^K$ such that H^{-1} exists on V and is differentiable.

Define

$$P : (\boldsymbol{z}, \boldsymbol{\pi}) \in \prod_{k\in[K]} \mathbb{R}_+^{S^k} \times \mathbb{R}_+^K \longmapsto \boldsymbol{y} \in \prod_{k\in[K]} \mathbb{R}_+^A$$

with

$$y_a^k = \left\{\begin{array}{ll} z_a^k & \text{if } a \in S^k \\ 0 & \text{otherwise.} \end{array}\right.$$

Let $(\vec{\boldsymbol{z}}, \boldsymbol{\pi}) = H^{-1}(\boldsymbol{0}, \boldsymbol{d}))$ for some $(\boldsymbol{0}, \boldsymbol{d}) \in V$. It is straightforward to check that $(P(\vec{\boldsymbol{z}}, \boldsymbol{\pi}), \boldsymbol{\pi})$ is a solution of the system above with d_0^k replaced by d^k, and thus that $P(\vec{\boldsymbol{z}}, \boldsymbol{\pi})$ satisfies the condition of Proposition 2. In other word, $\vec{\boldsymbol{e}}(\boldsymbol{d}) = P(H^{-1}(\boldsymbol{0}, \boldsymbol{d}))$ for $(\boldsymbol{0}, \boldsymbol{d}) \in V$. The map P is differentiable everywhere and H^{-1} is differentiable on V. It leads to the desired conclusion. □

4. Equilibrium and demand transfers: proof of Theorem 2

4.1. **A more general result.** Theorem 2 is a direct consequence of the following proposition. We define $\boldsymbol{d}_\delta$ the vector of demands after player i has transferred a part $\delta > 0$ of his demand to player j:

$$d_\delta^i = d^i - \delta, \qquad d_\delta^j = d^j + \delta, \quad \text{and} \quad d_\delta^k = d^k \quad \text{for } k \neq i, j.$$

Proposition 4. *Suppose that $|A| = 2$ or $K = 2$, and let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d})$ and $\vec{\boldsymbol{y}} = \vec{\boldsymbol{e}}(\boldsymbol{d}_\delta)$. Let $a \in A$.*
If $x_a^i \leq y_a^i$, then $x_a^i = y_a^i = 0$ and $x_a \leq y_a$.
If $x_a^j \geq y_a^j$, then $x_a^j = y_a^j = 0$ and $x_a \geq y_a$.
In case $|A| = 2$, we have moreover $(x_a^k - y_a^k)(x_a - y_a) \leq 0$ for $k \neq i, j$.

This proposition is proved in the next subsection. We first show two corollaries. They will be useful in the proof of Theorem 3. The second one is also used in the proof of Theorem 1.

Corollary 1. *Suppose that $|A| = 2$ or $K = 2$, and let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d})$ and $\vec{\boldsymbol{y}} = \vec{\boldsymbol{e}}(\boldsymbol{d}_\delta)$. Then,*

- $\text{supp}(\boldsymbol{y}^i) \subseteq \text{supp}(\boldsymbol{x}^i)$ *and* $\text{supp}(\boldsymbol{x}^j) \subseteq \text{supp}(\boldsymbol{y}^j)$.
- $\{a \in A : \ y_a < x_a\} \subseteq \text{supp}(\boldsymbol{x}^i)$ *and* $\{a \in A : \ y_a > x_a\} \subseteq \text{supp}(\boldsymbol{y}^j)$.

Proof. Let a be an arc in $\text{supp}(\boldsymbol{y}^i)$. We have $y_a^i \neq 0$. Proposition 4 implies then that $x_a^i > y_a^i$, which shows that $\text{supp}(\boldsymbol{y}^i) \subseteq \text{supp}(\boldsymbol{x}^i)$. The inclusion $\text{supp}(\boldsymbol{x}^j) \subseteq \text{supp}(\boldsymbol{y}^j)$ is proved similarly. This shows the first point.

Let a be such that $y_a < x_a$. Proposition 4 implies then that $x_a^i > y_a^i$, which shows that $a \in \text{supp}(\boldsymbol{x}^i)$. The other inclusion is proved similarly. This shows the second point. □

We consider now the "limit" case when the cost of an arc with no flow is equal to the marginal cost.

Corollary 2. *Suppose that $|A| = 2$ or $K = 2$, and let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d})$ and $\vec{\boldsymbol{y}} = \vec{\boldsymbol{e}}(\boldsymbol{d}_\delta)$.*

If $b \notin \mathrm{supp}(\boldsymbol{x}^j)$ is such that $c_b^j(x_b) = \min_{a\in A}(c_a^j(x_a) + x_a^j c_a^{j\prime}(x_a))$, then $b \in \mathrm{supp}(\boldsymbol{y}^j)$.

If $b \notin \mathrm{supp}(\boldsymbol{y}^i)$ is such that $c_b^i(y_b) = \min_{a\in A}(c_a^i(y_a) + y_a^j c_a^{i\prime}(y_a))$, then $b \in \mathrm{supp}(\boldsymbol{x}^i)$.

Proof. We only prove the first point, the proof of the second one being similar. If $x_b < y_b$, we have $b \in \mathrm{supp}(\boldsymbol{y}^j)$ because of Corollary 1. We can thus assume for the remaining of the proof that $x_b \geq y_b$. Suppose for a contradiction that $b \notin \mathrm{supp}(\boldsymbol{y}^j)$. Consider an arc $a \in \mathrm{supp}(\boldsymbol{y}^j)$. Proposition 4 gives that $x_a^j < y_a^j$. We have moreover

$$c_a^j(x_a) + x_a^j c_a^{j\prime}(x_a) \geq c_b^j(x_b) \geq c_b^j(y_b) \geq c_a^j(y_a) + y_a^j c_a^{j\prime}(y_a).$$

The first inequality is a consequence of the assumption on b, the second one is a consequence of the inequality $x_b \geq y_b$, the third one is a consequence of Proposition 2. Since c_a^j and $c_a^{j\prime}$ are increasing, we get that $x_a > y_a$.

Thus any arc $a \in \mathrm{supp}(\boldsymbol{y}^j)$ is such that $x_a > y_a$. Now, take such an arc α, i.e. such that $x_\alpha > y_\alpha$. Since the total demand is the same before and after the transfer, there must be an arc $\alpha' \neq \alpha$ such that $x_{\alpha'} < y_{\alpha'}$. According to Corollary 1, we have $\alpha' \in \mathrm{supp}(\boldsymbol{y}^j)$. This is in contradiction with the fact that such an arc must satisfy $x_{\alpha'} > y_{\alpha'}$. □

4.2. **Proof of Proposition 4.** We start with a technical lemma.

Lemma 2. *Let $\boldsymbol{d}^u, \boldsymbol{d}^v \in \mathbb{R}_+^K$ and let $\vec{\boldsymbol{u}} = \vec{\boldsymbol{e}}(\boldsymbol{d}^u)$ and $\vec{\boldsymbol{v}} = \vec{\boldsymbol{e}}(\boldsymbol{d}^v)$ for $k \in [K]$. Let a and b be two arcs such that $u_a \leq v_a$ and $u_b \geq v_b$. If k is a player such that $u_a^k < v_a^k$, then $u_b^k = v_b^k = 0$ or $u_b^k < v_b^k$.*

Proof. If $u_b^k = 0$, then $u_b^k \leq v_b^k$, with equality if and only if $u_b^k = v_b^k = 0$. We can thus suppose that $u_b^k > 0$. Proposition 2 gives $c_b^k(u_b) + u_b^k c_b^{k\prime}(u_b) \leq c_a^k(u_a) + u_a^k c_a^{k\prime}(u_a)$. Since $0 \leq u_a^k < v_a^k$, Proposition 2 gives $c_a^k(v_a) + v_a^k c_a^{k\prime}(v_a) \leq c_b^k(v_b) + v_b^k c_b^{k\prime}(v_b)$. These two equations together with the facts that c_a^k and $c_a^{k\prime}$ are increasing, $u_a \leq v_a$, and $u_a^k < v_a^k$ give that $c_b^k(u_b) + u_b^k c_b^{k\prime}(u_b) < c_b^k(v_b) + v_b^k c_b^{k\prime}(v_b)$. Finally, since $u_b \geq v_b$ and since c_b^k and $c_b^{k\prime}$ are increasing, we have $u_b^k < v_b^k$. □

Proof of Proposition 4. Let us first suppose that $\boldsymbol{x} = \boldsymbol{y}$. Since $d^i > d_\delta^i$, there is at least one arc a such that $x_a^i > y_a^i$. Lemma 2 with $\vec{\boldsymbol{u}} = \vec{\boldsymbol{y}}$ and $\vec{\boldsymbol{v}} = \vec{\boldsymbol{x}}$ implies that $y_b^i = x_b^i = 0$ or $y_b^i < x_b^i$ for all arcs $b \in A$. We get the desired conclusion for i. The proof of the conclusion for j is similar and omitted. Finally, in case $|A| = 2$, Lemma 2 with $\vec{\boldsymbol{u}} = \vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{v}} = \vec{\boldsymbol{y}}$ together with the fact that $d^k = d_\delta^k$ for $k \neq i, j$ shows that $x_a^k = y_a^k$ for both arcs a in A, leading to the desired inequality. We can thus assume for the remaining of the proof that $\boldsymbol{x} \neq \boldsymbol{y}$.

We first deal with the case $K = 2$. Since $\boldsymbol{x} \neq \boldsymbol{y}$, there is an arc $\alpha \in A$ such that $x_\alpha \neq y_\alpha$. Without loss of generality $x_\alpha < y_\alpha$, and there is an arc $\beta \in A$ such that $x_\beta > y_\beta$. We have $x_\alpha^\ell < y_\alpha^\ell$ for a player $\ell \in \{i, j\}$ and $x_\beta^{\ell'} > y_\beta^{\ell'}$ for a player $\ell' \in \{i, j\}$ (there are only two players).

For every arc b such that $x_b \geq y_b$, apply Lemma 2 with $a = \alpha$, $\vec{\boldsymbol{u}} = \vec{\boldsymbol{x}}$, $\vec{\boldsymbol{v}} = \vec{\boldsymbol{y}}$, and $k = \ell$. It gives that $x_b^\ell = y_b^\ell = 0$ or $x_b^\ell < y_b^\ell$. In particular when applied with $b = \beta$, we get $\ell \neq \ell'$. Since $x_b \geq y_b$, we have $x_b^{\ell'} \geq y_b^{\ell'}$.

For every arc b such that $x_b \leq y_b$, apply Lemma 2 with $a = \beta$, $\vec{\boldsymbol{u}} = \vec{\boldsymbol{y}}$, $\vec{\boldsymbol{v}} = \vec{\boldsymbol{x}}$, and $k = \ell'$. It gives that $x_b^{\ell'} = y_b^{\ell'} = 0$ and $y_b^{\ell'} < x_b^{\ell'}$. Since $x_b \leq y_b$, we have $x_b^\ell \leq y_b^\ell$.

Summing these inequalities over all arcs b, we get that $d^\ell \leq d_\delta^\ell$ and $d^{\ell'} \geq d_\delta^{\ell'}$, and we conclude that $\ell = j$ and $\ell' = i$.

We have proved that $x_a^i = y_a^i = 0$ or $x_a^i > y_a^i$ for every arc a such that $x_a \leq y_a$, and $x_a^i \geq y_a^i$ for every arc a such that $x_a \geq y_a$. It remains to prove that $x_a^i > y_a^i$ is strict for arcs a such that

$x_a > y_a$. Consider then such an arc a. Since $x_a^j \leq y_a^j$ and there are only two players, we have a strict inequality for player i too: $x_a^i > y_a^i$. A similar argument for player j leads to the desired conclusion.

Suppose that $|A| = 2$. Let α and β be the two arcs and suppose without loss of generality that $x_\alpha < y_\alpha$ and $x_\beta > y_\beta$. Applying Lemma 2 with $a = \alpha$, $b = \beta$, $\vec{\boldsymbol{u}} = \vec{\boldsymbol{x}}$, and $\vec{\boldsymbol{v}} = \vec{\boldsymbol{y}}$, we get that $x_\alpha^k \geq y_\alpha^k$ for all $k \neq j$ (using the fact the $d^k \geq d_\delta^k$ for such a player k). Since $x_\alpha < y_\alpha$, we get that $x_\alpha^j < y_\alpha^j$. For a player $k \neq i, j$, we have $d^k = d_\delta^k$. Thus $x_\beta^k \leq y_\beta^k$ for such a player. Note that we already get the last statement of the proposition.

Applying again Lemma 2 with $a = \alpha$, $b = \beta$, $\vec{\boldsymbol{u}} = \vec{\boldsymbol{x}}$, $\vec{\boldsymbol{v}} = \vec{\boldsymbol{y}}$, but this time for $k = j$, we get that $x_\beta^j = y_\beta^j = 0$ or $x_\beta^j < y_\beta^j$. This already gives the conclusion for player j. Since $x_\beta > y_\beta$, it implies that $x_\beta^i > y_\beta^i$. It remains to prove that the conclusion holds for player i. Let a be an arc such that $x_a^i \leq y_a^i$. We necessarily have $a = \alpha$. Applying Lemma 2, this times with $a = \beta$, $b = \alpha$, $\vec{\boldsymbol{u}} = \vec{\boldsymbol{y}}$, and $\vec{\boldsymbol{v}} = \vec{\boldsymbol{x}}$ allows then to conclude. □

5. Social cost at equilibrium when players have same cost functions: proof of Theorem 3

5.1. **Main steps of the proof.** The proof of Theorem 3 relies on two results: Corollary 1 and the following proposition, proved in the remaining of the section. We define $\boldsymbol{d}_\delta$ as in Section 4

$$d_\delta^i = d^i - \delta, \qquad d_\delta^j = d^j + \delta, \quad \text{and} \quad d_\delta^k = d^k \quad \text{for } k \neq i, j.$$

Throughout the section, the inequality $d^i \leq d^j$ is assumed.

Proposition 5. *Suppose that we are under the condition of Theorem 3 and that there exists $\delta > 0$ such that $\mathrm{supp}(\boldsymbol{e}^k(\boldsymbol{d})) = \mathrm{supp}(\boldsymbol{e}^k(\boldsymbol{d}_\delta))$ for both players $k = i$ and $k = j$. Then there exists $\eta > 0$ such that $h \mapsto Q(\vec{\boldsymbol{e}}(\boldsymbol{d}_h))$ is nonincreasing on $[0, \eta]$.*

If $\vec{\boldsymbol{e}}(\cdot)$ is differentiable, e.g. the cost functions are twice continuously differentiable (Theorem 1), then we can actually show that $h \mapsto Q(\vec{\boldsymbol{e}}(\boldsymbol{d}_h))$ is differentiable with a nonpositive derivative at 0.

The remaining of the section is devoted to the proof of this proposition. We finish this subsection by explaining how this proposition can be used to prove Theorem 3.

Proof of Theorem 3. Corollary 1 implies that when player i transfers a part of his demand to player j, the support of player i does not increase and the support of player j does not decrease. It implies that for $k = i$ and $k = j$, the set $\mathrm{supp}(\boldsymbol{e}^k(\boldsymbol{d}_h))$ changes a finite number of times when h goes from 0 to d^i. Hence, for almost all $h' \in [0, d^i]$, there exists some $\delta > 0$ (depending on h') such that $\mathrm{supp}(\boldsymbol{e}^k(\boldsymbol{d}_{h'})) = \mathrm{supp}(\boldsymbol{e}^k(\boldsymbol{d}_{h'+\delta}))$ for both $k = i$ and $k = j$. Thus, Proposition 5 implies that $h' \mapsto Q(\vec{\boldsymbol{e}}(\boldsymbol{d}_{h'}))$ is nonincreasing almost everywhere on $[0, d^i]$. Since $h' \mapsto Q(\vec{\boldsymbol{e}}(\boldsymbol{d}_{h'}))$ is continuous according to Proposition 3, the map $h' \mapsto Q(\vec{\boldsymbol{e}}(\boldsymbol{d}_{h'}))$ is nonincreasing on the whole interval $[0, d^i]$. □

5.2. **Technical lemmas.** Throughout this subsection, we assume that we are no longer in the player-specific cost function setting, i.e. we suppose that $c_a^1 = \cdots = c_a^K$ for all $a \in A$. The cost function attached to arc a is denoted c_a without superscript. We emphasize that the result of this subsection do not necessarily hold when we assume the costs to be player-specific.

We have divided this subsection into two parts. The first one deals with results regarding the supports of the strategies. The second one deals with the derivate of the social cost at equilibrium seen as a function of the transfer.

5.2.1. *Lemmas about the supports.* The following lemma has been proved by Orda et al. [1993]. It is stated here without proof.

Lemma 3. *Let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d})$. If $d^{k_1} \leq d^{k_2}$, then $x_a^{k_1} \leq x_a^{k_2}$ for every arc a.*

The next lemma deals with the case when the condition of Proposition 5 is satisfied and when moreover the supports of player i and j are identical. In this case, the social cost is not only nonincreasing, it is even constant.

Lemma 4. *Suppose that there exists $\delta \in (0, d^i]$ such that*

$$\operatorname{supp}(\boldsymbol{e}^i(\boldsymbol{d})) = \operatorname{supp}(\boldsymbol{e}^j(\boldsymbol{d})) = \operatorname{supp}(\boldsymbol{e}^i(\boldsymbol{d}_\delta)) = \operatorname{supp}(\boldsymbol{e}^j(\boldsymbol{d}_\delta)).$$

Then there exists $\eta > 0$ such that $h \mapsto Q(\vec{\boldsymbol{e}}(\boldsymbol{d}_h))$ is constant on $[0, \eta]$.

Proof. Let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d})$ and denote $\vec{\boldsymbol{y}}(h) = \vec{\boldsymbol{e}}(\boldsymbol{d}_h)$. Denote by S_0 be the common support: $S_0 = \operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{x}^j)$. Let

$$\pi^k = \min_{a \in A} \left(c_a(x_a) + x_a^k c_a(x_a) \right).$$

It is the marginal cost at equilibrium for player k when the demand is $\boldsymbol{d}$. Let $\eta = \min(\delta_0, \delta_1)$, where

$$\delta_0 = \min_{a \in S_0} \left(x_a^i c_a'(x_a) \right) \sum_{a \in S_0} \frac{1}{c_a'(x_a)} \qquad \text{and} \qquad \delta_1 = \left(\min_{b \notin S_0} c_b(x_b) - \pi^j \right) \sum_{a \in S_0} \frac{1}{c_a'(x_a)}.$$

η is well defined since $c_a'(x) > 0$ for every $x > 0$, and is nonnegative, according to Proposition 2. Moreover, Corollary 2 shows that $\eta > 0$.

Let $h \leq \eta$. We consider the strategy vector $\vec{\boldsymbol{z}}$ defined by

$$z_a^k = x_a^k + \frac{d_h^k - d^k}{\beta c_a'(x_a)} 1_{\{a \in S_0\}} \qquad \text{for every player } k$$

where $\beta = \sum_{a \in S_0} \frac{1}{c_a'(x_a)}$. The remaining of the proof consists in showing that $\vec{\boldsymbol{z}} = \vec{\boldsymbol{y}}(h)$. Since such a $\vec{\boldsymbol{z}}$ satisfies $z_a = x_a$ for all a (checking straightforward), it will show that $x_a = y_a(h)$ for all a, and in particular that $Q(\boldsymbol{x}) = Q(\boldsymbol{y}(h))$.

We first check that $\vec{\boldsymbol{z}}$ leads to feasible strategies. For each player k and arc a, we have $z_a^k \geq 0$ since $h \leq \delta_0$. We have moreover $\sum_{a \in A} z_a^k = d_h^k$ for each player k. Hence, we have feasible strategies.

We check now that $\vec{\boldsymbol{z}}$ is an equilibrium for $\boldsymbol{d}_h$, by checking that it satisfies the condition of Proposition 2. Consider first a player $k \in \{i, j\}$. For every arc $a \in S_0$, we have

$$c_a(z_a) + z_a^k c_a'(z_a) = c_a(x_a) + x_a^k c_a'(x_a) + \frac{d_h^k - d^k}{\beta} = \pi^k + \frac{d_h^k - d^k}{\beta}$$

Consider now an arc $a \notin S_0$. By definition of z_a^k, we have $z_a^k = x_a^k = 0$ (we still work with $k \in \{i, j\}$) and thus

$$c_a(z_a) = c_a(x_a) \geq \pi^j + \frac{h}{\beta} \geq \pi^k + \frac{d_h^k - d^k}{\beta},$$

where the first equality holds since $z_a = x_a$, the first inequality since $h \leq \delta_1$, and the last inequality for player i since $\pi^j \geq \pi^i$, according to Lemma 3. The condition of Proposition 2 is satisfied for every player $k \in \{i, j\}$. Consider now a player $k \neq \{i, j\}$. We have $d_h^k = d^k$ and thus $z_a^k = x_a^k$ for all $a \in A$. Hence, we have $c_a(z_a) + z_a^k c_a'(z_a) = \pi^k$ for $a \in \operatorname{supp}(\boldsymbol{z}^k)$ and $c_a(z_a) \geq \pi^k$ for $a \notin \operatorname{supp}(\boldsymbol{z}^k)$. Again, the Proposition 2 is satisfied, this time for the players $k \notin \{i, j\}$. Therefore, $\vec{\boldsymbol{z}}$ is an equilibrium for the demand $\boldsymbol{d}_h$.

By uniqueness of the equilibrium, $\vec{\boldsymbol{z}} = \vec{\boldsymbol{y}}(h)$. $\square$

The following lemma can be seen as a complement of Corollary 1.

Lemma 5. *Suppose that* $|A| = 2$ *or* $K = 2$. *Let* $\delta \in (0, d^i)$ *and denote* $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d})$, $\vec{\boldsymbol{y}} = \vec{\boldsymbol{e}}(\boldsymbol{d}_\delta)$. *Suppose that* $\operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^i) \neq \operatorname{supp}(\boldsymbol{x}^j) = \operatorname{supp}(\boldsymbol{y}^j)$. *If* $\boldsymbol{x} \neq \boldsymbol{y}$, *then*

$$\{a \in A, y_a < x_a\} = \operatorname{supp}(\boldsymbol{x}^i) \quad \textit{and} \quad \{a \in A, y_a > x_a\} = \operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i).$$

Proof. Assume that $\boldsymbol{x} \neq \boldsymbol{y}$, i.e. that there exists an arc a on which $x_a \neq y_a$ (and thus there are at least two such arcs, since the total demand remains constant).

Suppose first that $|A| = 2$. Corollary 1 gives that $\{a \in A : \ y_a < x_a\} \subseteq \operatorname{supp}(\boldsymbol{x}^i)$ and $\{a \in A : \ y_a > x_a\} \subseteq \operatorname{supp}(\boldsymbol{x}^j)$. Since $\boldsymbol{x} \neq \boldsymbol{y}$, then there exists an arc α with $y_\alpha < x_\alpha$ and an arc α' such that $y_{\alpha'} > x_{\alpha'}$. Since $\operatorname{supp}(\boldsymbol{x}^i) \subseteq \operatorname{supp}(\boldsymbol{x}^j)$ (Lemma 3), $|\operatorname{supp}(\boldsymbol{x}^j)| \leq 2$, and $\operatorname{supp}(\boldsymbol{x}^i) \neq \operatorname{supp}(\boldsymbol{x}^j)$, we get the conclusion.

Suppose now that $K = 2$. Again, there exists an arc α with $y_\alpha < x_\alpha$. Corollary 1 implies that $\alpha \in \operatorname{supp}(\boldsymbol{x}^i)$, and thus according to the assumption, we also have $\alpha \in \operatorname{supp}(\boldsymbol{y}^i)$. Lemma 3 implies that $\operatorname{supp}(\boldsymbol{x}^i) \subseteq \operatorname{supp}(\boldsymbol{x}^j)$ and $\operatorname{supp}(\boldsymbol{y}^i) \subseteq \operatorname{supp}(\boldsymbol{y}^j)$. Proposition 2 implies thus that

$$2c_a(x_a) + x_a c_a'(x_a) = 2c_\alpha(x_\alpha) + x_\alpha c_\alpha'(x_\alpha) \quad \text{for all } a \in \operatorname{supp}(\boldsymbol{x}^i),$$

by summing the two marginal costs for the two players i and j, and similarly that

$$2c_a(y_a) + y_a c_a'(y_a) = 2c_\alpha(y_\alpha) + y_\alpha c_\alpha'(y_\alpha) \quad \text{for all } a \in \operatorname{supp}(\boldsymbol{y}^i).$$

Since $y_\alpha < x_\alpha$ and since $u \mapsto 2c_a(u) + uc_a'(u)$ is increasing, we get $y_a < x_a$ for all $a \in \operatorname{supp}(\boldsymbol{x}^i)$. Combining this with the inclusion $\{a \in A : \ y_a < x_a\} \subseteq \operatorname{supp}(\boldsymbol{x}^i)$ given by Corollary 1, we get that $\{a \in A : \ y_a < x_a\} = \operatorname{supp}(\boldsymbol{x}^i)$.

Take now an arc a in $\operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^j) \setminus \operatorname{supp}(\boldsymbol{y}^i)$. It is necessarily such that $y_a^j = y_a \geq x_a = x_a^j$. Suppose for a contradiction that $y_a = x_a$, then Proposition 4 gives $x_a^j = y_a^j = 0$, which contradicts the way a has been taken. Thus $y_a > x_a$. Conversely, take an arc a such that $y_a > x_a$. It is not in $\operatorname{supp}(\boldsymbol{y}^i)$. Thus $y_a^j = y_a > 0$, which implies that $a \in \operatorname{supp}(\boldsymbol{y}^j)$. Since $\operatorname{supp}(\boldsymbol{y}^j) = \operatorname{supp}(\boldsymbol{x}^j)$, we get the conclusion. □

5.2.2. *Lemmas about the derivate of the social cost.* Given $\boldsymbol{z} = \mathbb{R}_+^A$ and an arc $a \in A$, we define the vector $\boldsymbol{z}^{-a} \in \mathbb{R}_+^{A \setminus \{a\}}$ by $\boldsymbol{z}^{-a} = (z_b)_{b \neq a}$. Then for every arc $a \in A$, we consider the function $Q_a : \mathbb{R}_+^{A \setminus \{a\}} \to \mathbb{R}_+$ defined by

$$Q_a(\boldsymbol{z}^{-a}) = \sum_{a' \neq a} z_b c_b(z_b) + \left(d - \sum_{b \neq a} z_b \right) c_a \left(d - \sum_{b \neq a} z_b \right),$$

where $d = \sum_{k \in [K]} d^k$ is the total demand.

Let $\boldsymbol{x} \in \mathbb{R}_+^A$ be such that $\sum_{a \in A} x_a = d$. We have then $Q_a(\boldsymbol{x}^{-a}) = Q(\vec{\boldsymbol{x}})$ and for any $b \neq a$

$$\frac{\partial Q_a}{\partial z_b}(\boldsymbol{x}^{-a}) = c_b(x_b) + x_b c_b'(x_b) - \big(c_a(x_a) + x_a c_a'(x_a)\big). \tag{2}$$

Denote $\vec{\boldsymbol{y}}(h) = \vec{\boldsymbol{e}}(\boldsymbol{d}_h)$.

Lemma 6. *Let* $h \in (0, d^i]$. *Suppose that* $\operatorname{supp}(\boldsymbol{y}^i(h)) \neq \operatorname{supp}(\boldsymbol{y}^j(h))$. *Let* $a \in \operatorname{supp}(\boldsymbol{y}^i(h))$ *and* $b \in \operatorname{supp}(\boldsymbol{y}^j(h)) \setminus \operatorname{supp}(\boldsymbol{y}^i(h))$. *We have then*

$$\frac{\partial Q_a}{\partial z_b}(\boldsymbol{y}^{-a}(h)) < 0.$$

Proof. Let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}(h)$. Since $a \in \operatorname{supp}(\boldsymbol{x}^i)$, we have $c_b(x_b) \geq c_a(x_a) + x_a^i c_a'(x_a) > c_a(x_a)$. Consider the set I_b of players k such that $b \in \operatorname{supp}(\boldsymbol{x}^k)$. The set I_b is nonempty and for every player $k \in I_b$ we have, according to Proposition 2, $c_b(x_b) + x_b^k c_b'(x_b) \leq c_a(x_a) + x_a^k c_a'(x_a)$. By summing over these players, we get

$$|I_b| c_b(x_b) + x_b c_b'(x_b) \leq |I_b| c_a(x_a) + \left(x_a - \sum_{k \notin I_b} x_a^k \right) c_a'(x_a) < |I_b| c_a(x_a) + x_a c_a'(x_a).$$

Since $|I_b| \geq 1$ and $c_b(x_b) > c_a(x_a)$, we have $c_b(x_b) + x_b c_b'(x_b) < c_a(x_a) + x_a c_a'(x_a)$, and thus, using Equation (2), we have $\dfrac{\partial Q_a}{\partial z_b}(\boldsymbol{x}^{-a}) < 0$. □

Lemma 7. *Suppose that $K = 2$. Let $h \in (0, d^i]$. There always exists an arc $a \in \operatorname{supp}(\boldsymbol{y}^i(h))$ such that*

$$\frac{\partial Q_a}{\partial z_b}(\boldsymbol{y}^{-a}(h)) \geq 0$$

for all $b \in \operatorname{supp}(\boldsymbol{y}^i(h)) \setminus \{a\}$.

Proof. Let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}(h)$. Pick an arc $a \in \arg\max\{c_\alpha(x_\alpha) : \alpha \in \operatorname{supp}(\boldsymbol{x}^i)\}$.

Let then $b \in \operatorname{supp}(\boldsymbol{x}^i) \setminus \{a\}$. Lemma 3 implies that $\operatorname{supp}(\boldsymbol{x}^i) \subseteq \operatorname{supp}(\boldsymbol{x}^j)$. Thus a and b are in $\operatorname{supp}(\boldsymbol{x}^j)$. Summing the marginal costs for players i and j, we get

$$2c_a(x_a) + x_a c_a'(x_a) = 2c_b(x_b) + x_b c_b'(x_b).$$

According to the definition of a, we have $c_a(x_a) \geq c_b(x_b)$ and then

$$c_a(x_a) + x_a c_a'(x_a) \leq c_b(x_b) + x_b c_b'(x_b).$$

Thus, using Equation (2), we have $\dfrac{\partial Q_a}{\partial z_b}(\boldsymbol{x}^{-a}) \geq 0$. □

5.3. **Proof of Proposition 5.**

Proof of Proposition 5. Let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e}}(\boldsymbol{d})$ and denote $\vec{\boldsymbol{y}}(h) = \vec{\boldsymbol{e}}(\boldsymbol{d}_h)$.

Suppose first that $|A| = 2$. If we are under the condition of Lemma 4, the conclusion is immediate. We can thus assume that $\operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^i(\delta)) \neq \operatorname{supp}(\boldsymbol{x}^j) = \operatorname{supp}(\boldsymbol{y}^j(\delta))$. Since there are only two arcs, there is an arc $a \in \operatorname{supp}(\boldsymbol{x}^i)$ and an arc $b \in \operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i)$. We have $Q(\vec{\boldsymbol{y}}(h)) = Q_a(y_b(h))$ (there are only two arcs in A). A first application of Corollary 1 shows that $\operatorname{supp}(\boldsymbol{y}^i(h)) \subseteq \operatorname{supp}(\boldsymbol{x}^i)$. A second application shows that $\operatorname{supp}(\boldsymbol{y}^i(\delta)) \subseteq \operatorname{supp}(\boldsymbol{y}^i(h))$. Since it is assumed that $\operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^i(\delta))$ for all $h \in [0, \delta]$, we get that $\operatorname{supp}(\boldsymbol{y}^i(h) = \operatorname{supp}(\boldsymbol{x}^i)$ for all $h \in [0, \delta]$. Similarly, $\operatorname{supp}(\boldsymbol{y}^j(h) = \operatorname{supp}(\boldsymbol{x}^j)$ for all $h \in [0, \delta]$

Lemma 5 shows that the total flow on b increases when h increases (b is in $\operatorname{supp}(\boldsymbol{y}^j(h)) \setminus \operatorname{supp}(\boldsymbol{y}^i(h))$). The map $h \mapsto y_b(h)$ is thus an increasing map on $[0, \delta]$. According to Lemma 6, we have

$$\frac{\partial Q_a}{\partial z_b}(y_b(h)) < 0.$$

Thus Q_a is decreasing on $[y_b(0), y_b(\delta)]$. Therefore the map $h \mapsto Q(\vec{\boldsymbol{y}}(h))$ is decreasing on $[0, \delta]$.

Suppose now that $K = 2$ and that the cost functions are twice continuously differentiable. If we are under the condition of Lemma 4, the conclusion is immediate. We can thus assume that $\operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^i(\delta)) \neq \operatorname{supp}(\boldsymbol{x}^j) = \operatorname{supp}(\boldsymbol{y}^j(\delta))$. Again, $\operatorname{supp}(\boldsymbol{y}^i(h))$ and $\operatorname{supp}(\boldsymbol{y}^j(h))$ are constant for all $h \in [0, \delta]$.

Choose $h_0, h \in (0,\delta)$, with $h \neq h_0$. Let $a \in \operatorname{supp}(\boldsymbol{y}^i(h_0))$ as in Lemma 7. Lemma 5 shows that $y_b(h) - y_b(h_0)$ and $h - h_0$ have opposite signs when $b \in \operatorname{supp}(\boldsymbol{y}^i(h_0)) \setminus \{a\}$ and same sign when $b \in \operatorname{supp}(\boldsymbol{y}^j(h_0)) \setminus \operatorname{supp}(\boldsymbol{y}^i(h_0))$. Thus,

$$\sum_{b \in \operatorname{supp}(\boldsymbol{y}^i(h_0)) \setminus \{a\}} \frac{\partial Q_a}{\partial z_b}\left(\boldsymbol{y}^{-a}(h_0)\right) \frac{y_b(h) - y_b(h_0)}{h - h_0} \leq 0$$

for all $h \in (0,\delta)$, and

$$\sum_{b \in \operatorname{supp}(\boldsymbol{y}^j(h_0)) \setminus \operatorname{supp}(\boldsymbol{y}^i(h_0))} \frac{\partial Q_a}{\partial z_b}\left(\boldsymbol{y}^{-a}(h_0)\right) \frac{y_b(h) - y_b(h_0)}{h - h_0} \leq 0$$

for all $h \in (0,\delta)$ (with the help of Lemma 6). This sum over arcs b not in $\operatorname{supp}(\boldsymbol{y}^j(h_0))$ is zero since $y_b(h) = y_b(h_0) = 0$. Besides, the cost functions being differentiable and since there are only two players, we can apply Theorem 1: $h \mapsto \vec{\boldsymbol{y}}(h)$ is differentiable in a neighborhood of h_0. Using the above inequalities, we get that the derivative of $h \mapsto Q(\vec{\boldsymbol{y}}(h))$ is nonpositive at h_0:

$$\sum_{b \neq a} \frac{\partial Q_a}{\partial z_b}(\boldsymbol{y}^{-a}(h_0)) y_b'(h_0) = \lim_{h \to h_0} \sum_{b \neq a} \frac{\partial Q_a}{\partial z_b}(\boldsymbol{y}^{-a}(h_0)) \frac{y_b(h) - y_b(h_0)}{h - h_0} \leq 0.$$

Since this is true for any $h_0 \in (0,\delta)$, we get the conclusion: $h \mapsto Q(\vec{\boldsymbol{y}}(h))$ is nonincreasing on $[0,\delta]$ (using the continuity of $h \mapsto \vec{\boldsymbol{y}}(h)$ ensured by Proposition 3 to get the conclusion for the boundary of $[0,\delta]$). $\square$

6. Discussion

6.1. $\vec{\boldsymbol{e}}(\cdot)$ is not differentiable everywhere. The application $\vec{\boldsymbol{e}}(\cdot)$ is not differentiable everywhere as shown by the following example, inspired from Hall [1978]. Consider a parallel-link graph with two arcs a and b and only one player with demand d. Suppose that the cost functions are

$$c_a(x) = x + 1 \quad \text{and} \quad c_b(x) = x.$$

A direct calculation gives that at equilibrium $x_a = 0$ and $x_b = d$ when $0 \leq d \leq \frac{1}{2}$, and $x_a = \frac{2d-1}{4}$, $x_b = \frac{2d+1}{4}$ when $d \geq \frac{1}{2}$. Hence $\vec{\boldsymbol{e}}$ is not differentiable at the point $d = \frac{1}{2}$.

6.2. When there are three arcs and three players. Theorems 3 is not valid when there are three arcs and three players. We introduce the example of Huang [2011], for which the social cost at equilibrium increases after a transfer.

Consider a parallel-link graph with three arcs a, b, and c, and three players 1, 2, and 3. Suppose that the players have the same cost functions

$$c_a(x) = 20x + 5000, \quad c_b(x) = x^2 + 500, \quad \text{and} \quad c_c(x) = x^{11}.$$

When the vector of demand is $\boldsymbol{d} = (0.1, 20.9, 200)$, the (rounded) flows at equilibrium are in the following table and the equilibrium cost is 1 558 627.

Arc		a	b	c
Flow	player 1	0	0	0.1
	player 2	0	20.18	0.72
	player 3	152.50	46.38	1.12
Total flow		152.50	66.55	1.95

After the transfer of $\delta = 0.1$ from player 1 to player 2, we have demands $\boldsymbol{d}_\delta = (0, 21, 200)$. The (rounded) flows at equilibrium are in the following table and the social cost at equilibrium is 1 558 633. In particular, the cost has increased after the transfer.

Arc		a	b	c
	player 1	0	0	0
Flow	player 2	0	20.24	0.76
	player 3	152.49	46.33	1.18
Total flow		152.49	66.57	1.94

Moreover, the part regarding players that keep the same demand in Proposition 4 (more general result than Theorem 2) does not hold either, since $(y_a^3 - x_a^3)(y_a - x_a) > 0$, where $\vec{\boldsymbol{x}}$ (resp. $\vec{\boldsymbol{y}}$) is the equilibrium flow before (resp. after) the transfer. However Theorem 2 still holds: on each arc the flow of player 1 decreases or remains constant equal to zero, and the flow of player 2 increases or remains constant equal to zero.

6.3. **When we allow player-specific cost functions.** Another question is whether Theorem 3 remains valid when we allow player-specific cost functions. The answer is 'no', as shown by the following example.

Consider a parallel-link graph with two arcs a and b, and two players 1 and 2. Let the cost on arc b for player 1 (resp. on arc a for player 2) be prohibitively high, in such a way that at equilibrium, for every repartition of the demand, player 1 (resp. 2) puts all his demand on arc a (resp. b). Let the costs be $c_a^1(x) = 2x$, $c_b^2(x) = x$. If $d^1 = 3$ and $d^2 = 2$, the social cost at equilibrium is 8, while after a transfer of 1, i.e. if $d^1 = 4$ and $d^2 = 1$, the social cost at equilibrium is 9. The result of Theorem 3 does not hold if we allow player-specific costs.

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Université Paris-Est, CERMICS (ENPC), 6-8 Avenue Blaise Pascal, Cité Descartes, 77455 Marne-la-Vallée, France